Example 1 Real Roots

Use the quadratic formula to solve each quadratic equation. Where necessary, round to the nearest hundredth.
Verify graphically using technology.

a) $2x^2 + 9x + 6 = 0$

b) $4x^2 - 12x = -9$

Solution

a) For $2x^2 + 9x + 6 = 0$, $a = 2$, $b = 9$, and $c = 6$.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-9 \pm \sqrt{9^2 - 4(2)(6)}}{2(2)} \\
&= \frac{-9 \pm \sqrt{81 - 48}}{4} \\
&= \frac{-9 \pm \sqrt{33}}{4}
\end{aligned}
$$

The exact roots are $\frac{-9 + \sqrt{33}}{4}$ and $\frac{-9 - \sqrt{33}}{4}$.

You can also express the answers as approximate roots.

$$x = \frac{-9 + \sqrt{33}}{4} \doteq -0.81 \quad \text{or} \quad x = \frac{-9 - \sqrt{33}}{4} \doteq -3.69$$

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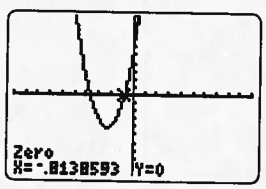
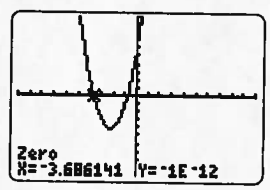
(-9+√(33))/4
          -.8138593384
(-9-√(33))/4
          -3.686140662
```

The approximate roots are -0.81 and -3.69, to the nearest hundredth.

Use the Zero operation of a graphing calculator to verify that the roots are the zeros of the related quadratic relation $y = 2x^2 + 9x + 6$.





Technology Tip

When a graphing calculator displays a result such as −1E−12, it represents a number in scientific notation.

$-1E{-}12 = -1 \times 10^{-12}$

b) First, write $4x^2 - 12x = -9$ in the form $ax^2 + bx + c = 0$.

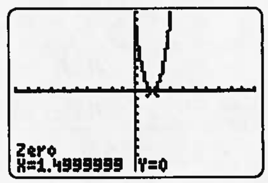
$4x^2 - 12x + 9 = 0$

For $4x^2 - 12x + 9 = 0$, $a = 4$, $b = -12$, and $c = 9$.

I can see that the trinomial on the left side of the equation is a perfect square, so I could have solved by factoring.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} \\ &= \frac{12 \pm \sqrt{144 - 144}}{8} \\ &= \frac{12 \pm \sqrt{0}}{8} \\ &= \frac{12}{8} \\ &= \frac{3}{2} \end{aligned}$$

The root is $\frac{3}{2}$ or 1.5.



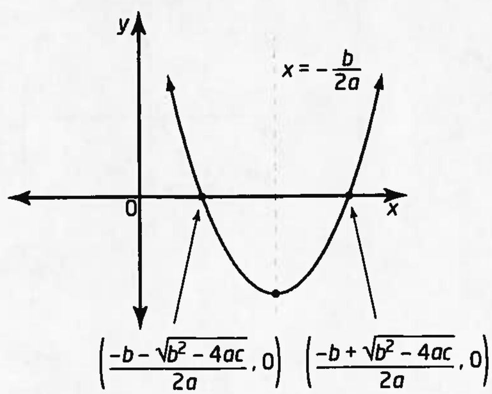
When the x-intercepts are known, you can find the x-coordinate of the vertex by finding the midpoint of the line segment connecting the x-intercepts.

Use the two x-intercepts from the quadratic formula.

$$\begin{aligned} x &= \frac{\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}}{2} \\ &= \frac{\dfrac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}}{2} \\ &= \frac{\dfrac{-2b}{2a}}{2} \\ &= \frac{-2b}{4a} \\ &= -\frac{b}{2a} \end{aligned}$$

The square root expressions have a zero sum.

$\sqrt{b^2 - 4ac} - \sqrt{b^2 - 4ac} = 0$



The x-coordinate of the vertex is $-\frac{b}{2a}$.

This also gives the equation of the axis of symmetry, $x = -\frac{b}{2a}$.

Example 2 Use the Quadratic Formula to Sketch a Parabola

Find the x-intercepts, the vertex, and the equation of the axis of symmetry of the quadratic relation $y = -5x^2 + 8x - 3$. Sketch the parabola.

Solution

To find the x-intercepts, let $y = 0$ and use the quadratic formula to solve the quadratic equation.

For $-5x^2 + 8x - 3 = 0$, $a = -5$, $b = 8$, and $c = -3$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-8 \pm \sqrt{8^2 - 4(-5)(-3)}}{2(-5)} \\ &= \frac{-8 \pm \sqrt{64 - 60}}{-10} \\ &= \frac{-8 \pm \sqrt{4}}{-10} \\ &= \frac{-8 \pm 2}{-10} \end{aligned}$$

Therefore,

$$x = \frac{-8 - 2}{-10} \quad \text{or} \quad x = \frac{-8 + 2}{-10}$$

$$= 1 \qquad\qquad = \frac{3}{5} \text{ or } 0.6$$

The x-intercepts are 0.6 and 1.

To find the x-coordinate of the vertex, use $x = -\frac{b}{2a}$.

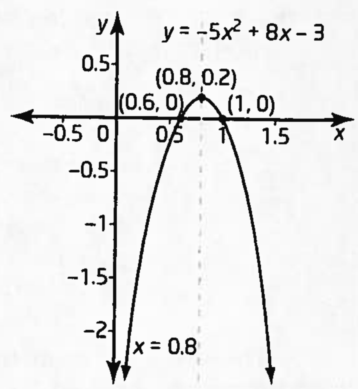
$$\begin{aligned} x &= -\frac{8}{2(-5)} \\ &= -\frac{8}{-10} \\ &= \frac{4}{5} \text{ or } 0.8 \end{aligned}$$

I can check this one mentally. The x-coordinate of the midpoint of the line segment connecting the x-intercepts, 0.6 and 1, is 0.8.

Substitute $x = 0.8$ into $y = -5x^2 + 8x - 3$ to find the y-coordinate of the vertex.

$$\begin{aligned} y &= -5(0.8)^2 + 8(0.8) - 3 \\ &= -3.2 + 6.4 - 3 \\ &= 0.2 \end{aligned}$$

The coordinates of the vertex are (0.8, 0.2).
The axis of symmetry has equation $x = 0.8$.



Example 3 Connect a Parabola and No Real Roots

A parabola has equation $y = (x - 2)^2 + 3$.

a) State the coordinates of the vertex, the equation of the axis of symmetry, and the direction of opening.

b) Determine the x-intercepts. Verify using the quadratic formula.

c) Sketch the parabola.

Solution

a) The vertex is (2, 3). The equation of the axis of symmetry is $x = 2$. The parabola opens upward, since a is positive.

b) Since the vertex of the parabola is above the x-axis and it opens upward, it has no x-intercepts. This can be verified using the quadratic formula after expanding and simplifying the original equation.

$$\begin{aligned} y &= (x - 2)^2 + 3 \\ &= x^2 - 4x + 4 + 3 \\ &= x^2 - 4x + 7 \end{aligned}$$

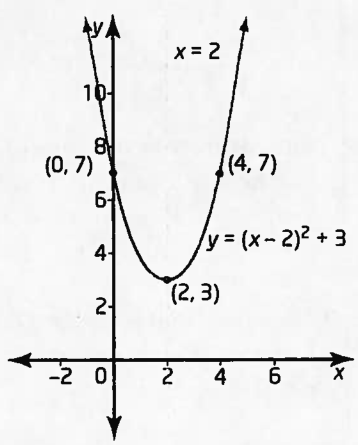
Let $y = 0$ and use the quadratic formula with $a = 1$, $b = -4$, and $c = 7$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(\quad) \pm \sqrt{(\quad)^2 - 4(\ )(\ )}}{2(\ )} \\ &= \frac{4 \pm \sqrt{16 - 28}}{2} \\ &= \frac{4 \pm \sqrt{-12}}{2} \end{aligned}$$

Since the square root of a negative number is not a real number, there are no real roots.
Therefore, the parabola has no x-intercepts.

c) Let $x = 0$. A second point on the curve is (0, 7).

Looking at the x-coordinate of this point, I know that it is located 2 units to the left of the axis of symmetry. So, its partner is located 2 units to the right of the axis of symmetry and has the same y-coordinate.



Then, due to symmetry, the partner point on the parabola is (4, 7).

Example 4 Path of a Basketball

The path of a basketball after it is thrown from a height of 1.5 m above the ground is given by the equation $h = -0.25d^2 + 2d + 1.5$, where h is the height, in metres, and d is the horizontal distance, in metres.

a) How far has the ball travelled horizontally, to the nearest tenth of a metre, when it lands on the ground?

b) Find the horizontal distance when the basketball is at a height of 4.5 m above the ground.

Solution

a) When the basketball lands on the ground, the height is 0 m.
Let $h = 0$.
For $-0.25d^2 + 2d + 1.5 = 0$, $a = -0.25$, $b = 2$, and $c = 1.5$.

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4(-0.25)(1.5)}}{2(-0.25)}$$

$$= \frac{-2 \pm \sqrt{4 + 1.5}}{-0.5}$$

$$= \frac{-2 \pm \sqrt{5.5}}{-0.5}$$

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(-2+√(5.5))/-0.5
        -.6904157598
(-2-√(5.5))/-0.5
         8.69041576
```

So, $d \doteq -0.7$ or $d \doteq 8.7$.

Since d represents distance, it must be positive.
The basketball has travelled a horizontal distance of about 8.7 m when it lands on the ground.

b) Let $h = 4.5$.

$-0.25d^2 + 2d + 1.5 =$

$-0.25d^2 + 2d - 3 = 0$

For $-0.25d^2 + 2d - 3 = 0$, $a = -0.25$, $b = 2$, and $c = -3$.

$$
\begin{aligned}
d &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{- \quad \pm \sqrt{\quad ^2 - 4(\qquad)(\quad)}}{2(\qquad)} \\
&= \frac{-2 \pm \sqrt{4 - 3}}{-0.5} \\
&= \frac{-2 \pm \sqrt{1}}{-0.5} \\
&= \frac{-2 \pm 1}{-0.5}
\end{aligned}
$$

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(-2+1)/-0.5
                2
(-2-1)/-0.5
                6
```

So, $d = 2$ or $d = 6$.

The basketball will be at a height of 4.5 m twice along its parabolic path: on the way up at a horizontal distance of 2 m and on the way down at a horizontal distance of 6 m.

I need to express the equation in the form $ax^2 + bx + c = 0$.

Key Concepts

- A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, can be solved for x using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The x-coordinate of the vertex of a parabola is $-\frac{b}{2a}$, and the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

Solving Quadratic Equations: $ax^2 + bx + c = 0$

Method		Comments
graphing	always	The solutions will not always be exact: this is best used only when an approximate answer is needed.
factoring	sometimes	Use when $c = 0$ or when factors are easily found.
completing the square	always	This is best used for equations of the form $x^2 + bx + c = 0$, where b is an even number.
quadratic formula	always	This method always gives exact solutions, but in some cases the other methods are easier to use.
computer algebra system (CAS)	always	Use the solve (or factor) function of the CAS.